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Abstract

Let (Z_n) be a supercritical branching process in an independent and identically distributed random environment $\zeta = (\zeta_0, \zeta_1, \ldots)$, and let W be the limit of the normalized population size $Z_n/\mathbb{E}(Z_n|\zeta)$. We show a necessary and sufficient condition for the existence of weighted moments of W of the form $\mathbb{E}W^\alpha \ell(W)$, where $\alpha \geq 1$, ℓ is a positive function slowly varying at ∞. In the Galton-Watson case, the results improve the corresponding ones of Bingham and Doney (1974) and Alsmeyer and Rösler (2004).

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1 Introduction and main results

For a Galton-Watson process (Z_n) with offspring mean $m = \mathbb{E}Z_1 \in (1, \infty)$, the moments of $W = \lim Z_n/m^n$ have been studied by many authors: see for example Harris (1963), Athreya and Ney (1972), Bingham and Doney (1974), Alsmeyer and Rösler (2004), Iksanov (2006). Of particular interest is the following comparison theorem about weighted moments of W and Z_1, first proved by Bingham and Doney (1974) via Tauberian theorems: when $\alpha > 1$ is not an integer and ℓ is a positive function slowly varying at ∞, $\mathbb{E}W^\alpha \ell(W) < \infty$ if and only if $\mathbb{E}Z_1^\alpha \ell(Z_1) < \infty$. Alsmeyer and Rösler (2004) showed that the equivalence remains true when α is not of the form 2^n for some integer $n \geq 1$, by a nice martingale argument. In this paper, we show that the equivalence is always true whenever $\alpha > 1$, and that a similar result holds for a branching process in an independent and identically distributed random environment. Our approach is a refinement of the martingale argument of Alsmeyer and Rösler (2004). We mention that the adaptation of the argument to the random environment case is not evident; actually in this case the study of the existence of the moments of order α is already delicate, see for example Afanasyev (2001, Sect. 3) and Guivarc'h and Liu (2001) where this problem has been considered.

Let $\zeta = (\zeta_0, \zeta_1, \ldots)$ be a sequence of independent and identically distributed (i.i.d.) random variables, taking values in some space Θ, whose realization corresponds to a sequence of probability distributions on $\mathbb{N} = \{0, 1, 2, ...\}$:

$$p(\zeta_n) = \{p_i(\zeta_n) : i \geq 0\}, \text{ where } p_i(\zeta_n) \geq 0, \ \sum_{i=0}^{\infty} p_i(\zeta_n) = 1. \tag{1.1}$$

A branching process $(Z_n)_{n\geq 0}$ in the random environment ζ (BPRE) is a family of time-inhomogeneous branching processes (see e.g. [5, 6, 7]): given the environment ζ, the process $(Z_n)_{n\geq 0}$ acts as a Galton-Watson process in varying environments with offspring distributions $p(\zeta_n)$ for particles in the nth generation, $n \geq 0$. By definition,

$$Z_0 = 1 \qquad \text{and} \qquad Z_{n+1} = \sum_{u \in T_n} X_u \quad \text{for} \quad n \geq 0, \tag{1.2}$$

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where conditioned on ζ, $\{X_u : |u| = n\}$ are integer-valued random variables with common distribution $p(\zeta_n)$; all the random variables X_u, indexed by finite sequences of integers u, are conditionally independent of each other. Here T_n denotes the set of all individuals of generation n, marked by sequences u of positive integers of length $|u| = n$: as usual, the initial particle is denoted by the empty sequence $\o$ (of length 0); if $u \in T_n$, then $ui \in T_{n+1}$ if and only if $1 \leq i \leq X_u$. The classical Galton-Watson process corresponds to the case where all ζ_n are the same constant.

Let $(\Gamma, \mathbb{P}_\zeta)$ be the probability space under which the process is defined when the environment ζ is given. Therefore under $\mathbb{P}_\zeta$, the random variables X_u are independent of each other, and have the common law $p(\zeta_n)$ if $|u| = n$. The probability $\mathbb{P}_\zeta$ is usually called *quenched law*. The total probability space can be formulated as the product space $(\Theta^{\mathbb{N}} \times \Gamma, \mathbb{P})$, where $\mathbb{P} = \mathbb{P}_\zeta \otimes \tau$ in the sense that for all measurable and positive function g, we have

$$\int g d\mathbb{P} = \int \int g(\zeta, y) d\mathbb{P}_\zeta(y) d\tau(\zeta),$$

where τ is the law of the environment ζ. The total probability $\mathbb{P}$ is called *annealed law*. The quenched law $\mathbb{P}_\zeta$ may be considered to be the conditional probability of the annealed law $\mathbb{P}$ given ζ. The expectation with respect to $\mathbb{P}_\zeta$ (resp. $\mathbb{P}$) will be denoted $\mathbb{E}_\zeta$ (resp. $\mathbb{E}$).

For $n \geq 0$, write

$$m_n = \sum_{i=0}^{\infty} i p_i(\zeta_n), \quad \Pi_0 = 1 \quad \text{and} \quad \Pi_n = m_0 \cdots m_{n-1} \text{ if } n \geq 1. \tag{1.3}$$

Then $\mathbb{E}_\zeta X_u = m_n$ if $|u| = n$, and $\mathbb{E}_\zeta Z_n = \Pi_n$ for each n.

We consider the supercritical case where

$$\mathbb{E} \ln m_0 \in (0, \infty].$$

It is well-known that under $\mathbb{P}_\zeta$,

$$W_n = \frac{Z_n}{\Pi_n} \qquad (n \geq 0)$$

forms a nonnegative martingale with respect to the filtration

$$\mathcal{E}_0 = \{\emptyset, \Omega\} \qquad \text{and} \qquad \mathcal{E}_n = \sigma\{\zeta, X_u : \ |u| < n\} \quad \text{for } n \geq 1.$$

It follows that $(W_n, \mathcal{E}_n)$ is also a martingale under $\mathbb{P}$. Let

$$W = \lim_{n\to\infty} W_n \qquad \text{and} \qquad W^* := \sup_{n \geq 0} W_n, \tag{1.4}$$

where the limit exists a.s. by the martingale convergence theorem, and $\mathbb{E}W \leq 1$ by Fatou's lemma.

We are interested in asymptotic properties of W. Recall that in [14], Guivarc'h and Liu gave a necessary and sufficient condition for the existence of moments of W of order $\alpha > 1$:

Lemma 1.1 *([14, Theorem 3]) Let (Z_n) be a supercritical branching process in an i.i.d. random environment. Let $\alpha > 1$. Then $0 < \mathbb{E}W^\alpha < \infty$ if and only if $\mathbb{E}m_0^{-(\alpha-1)} < 1$ and $\mathbb{E}W_1^\alpha < \infty$.*

This result suggests that under a moment condition on m_0, W_1 and W have similar tail behavior. In the following, we shall establish comparison theorems between weighted moments of W_1 and W.

Recall that a positive and measurable function ℓ defined on $[0, \infty)$ is called slowly varying at ∞ if $\lim_{x\to\infty} \ell(\lambda x)/\ell(x) = 1$ for all $\lambda > 0$. (Throughout this paper, the term "positive" is used in the wide sense.) By the representation theorem (see [11, Theorem 1.3.1]), any slowly varying function ℓ is of the form

$$\ell(x) = c(x) \exp\left(\int_{a_0}^{x} \epsilon(t) dt/t\right), \qquad x > a_0, \tag{1.5}$$

where $a_0 \geq 0$, $c(\cdot)$ and $\epsilon(\cdot)$ are measurable with $c(x) \to c$ for some constant $c \in (0, \infty)$, and $\epsilon(x) \to 0$, as $x \to \infty$. The value of a_0 and those of $\ell(x)$ on $[0, a_0]$ will not be important; we always assume that ℓ is bounded on compact sets of $[0, \infty)$. For convenience, we often take $a_0 = 1$.

We search for conditions under which W has weighted moments of the form $\mathbb{E}W^\alpha \ell(W)$, where $\alpha \geq 1$, $\ell \geq 0$ is a function slowly varying at ∞. Notice that the function $c(x)$ in the representation of $\ell(x)$ has no influence on the finiteness of the moments, so that we can suppose without loss of generality that $c(x) = 1$.

We first consider the case where $\alpha > 1$. As usual, for a set A, we write Int$A$ for its interior.

Theorem 1.1 *Let $\alpha \in \mathrm{Int}\{a > 1 : \mathbb{E}m_0^{1-a} < 1\}$ and $\ell : [0,\infty) \mapsto [0,\infty)$ be a function slowly varying at ∞. Then the following assertions are equivalent:*

(a) $\mathbb{E}W_1^\alpha \ell(W_1) < \infty$ *;*

(b) $\mathbb{E}W^{*\alpha} \ell(W^*) < \infty$ *;*

(c) $0 < \mathbb{E}W^\alpha \ell(W) < \infty$ *.*

The result is sharp even for the classical Galton-Watson process (where ζ_n are the same constant): in this case, it improves the corresponding result of Bingham and Doney (1974) in the sense that they needed the additional assumption that $\ell(x) = \int_1^x \ell_0(x)/x dx$ for some function ℓ_0 slowly varying at ∞ (which is equivalent to the hypothesis that the function $\epsilon(.)$ in (1.5) is positive and slowly varying at ∞) when α is an integer. Alsmeyer and Rösler (2004) showed that this additional condition can be removed if α is not a dyadic power; our result shows that it can be removed for all α and that the same conclusion holds even in the random environment case.

We now consider the case where $\alpha = 1$, where the situation is different as already shown by Bingham and Doney (1974) in the Galton-Watson case.

For a measurable function $\ell : [0,\infty) \mapsto [0,\infty)$, we set

$$\hat{\ell}(x) = \begin{cases} \int_1^x \frac{\ell(t)}{t} dt & \text{if } x > 1; \\ 0 & \text{if } x \le 1. \end{cases} \tag{1.6}$$

We essentially deal with the case where ℓ is concave, which covers the case of slowly varying functions considered by Bingham and Doney (1974). (cf. Corollary 1.1 below)

Theorem 1.2 *Let $\ell : [0,\infty) \mapsto [0,\infty)$ be concave on $[a_0,\infty)$ for some $a_0 \ge 0$. If $\mathbb{E}m_0^{-1} < 1$ and $\mathbb{E}W_1 \hat{\ell}(W_1) < \infty$, then*

$$\mathbb{E}W^* \ell(W^*) < \infty \qquad \text{and} \qquad \mathbb{E}W \ell(W) < \infty .$$

Moreover, in the case where ℓ is also slowly varying at ∞, the moment condition $\mathbb{E}m_0^{-1} < 1$ can be relaxed to $\mathbb{E}m_0^{-\delta_0} < \infty$ for some $\delta_0 > 0$.

Notice that when ℓ is positive and concave on $[a_0,\infty)$, ℓ must be increasing, since otherwise there would exist $x_0 > a_0$ such that $\ell'(x_0) < 0$, where ℓ' denotes the left (or right) derivative (which is well-defined), so that for $x > x_0$, $\ell(x) \le \ell(x_0) + \ell'(x_0)(x - x_0) \to -\infty$ as $x \to \infty$. If $\lim_{x\to\infty} \ell(x) = \infty$, the conclusion for $\mathbb{E}W\ell(W)$ was obtained by Iksanov (2006) in the Galton-Watson case; a similar result was shown by Iksanov and Rösler (2006) for branching random walks. If $\lim_{x\to\infty} \ell(x) = c \in (0,\infty)$, the conclusion about W^* leads to a new proof for the non-degeneration of W: cf. the comments before Theorem 1.3 below.

As a corollary of Theorem 1.2, we obtain:

Corollary 1.1 *Let $\ell : [0,\infty) \mapsto [0,\infty)$ be nondecreasing and slowly varying at ∞, such that $\ell(x) = \int_1^x \ell_0(t)dt/t$ for some function $\ell_0 \ge 0$ slowly varying at ∞. Assume that $\mathbb{E}m_0^{-\delta_0} < \infty$ for some $\delta_0 > 0$. If $\mathbb{E}W_1 \hat{\ell}(W_1) < \infty$, then*

$$\mathbb{E}W^* \ell(W^*) < \infty \qquad \text{and} \qquad \mathbb{E}W \ell(W) < \infty .$$

Corollary 1.1 extends the sufficiency of Theorem 7 of Bingham and Doney (1974) where the classical Galton-Watson process was considered; see also Corollary 2.3 of Alsmeyer and Rösler (2004). As information, we mention that Alsmeyer and Iksanov (2009) gave sufficient conditions for $\mathbb{E}Wb(\ln^+ W) < \infty$ in the case of branching random walks, where $b > 0$ is a regular function of order $a > 0$. Notice that their theorem does not cover our Corollary 1.1. For example, our Corollary 1.1 applies for $\ell(x) = \ln\ln x$ (which corresponds to (1.5) with $c(x) = 1$ and $\epsilon(t) = (\ln t \cdot \ln\ln t)^{-1}$) to obtain a sufficient condition for $\mathbb{E}W\ell(W) < \infty$; but this result is not covered by Theorem 1.4 of Alsmeyer & Iksanov (2009), as $b(x) = \ell(e^x) = \ln x$ is slowly varying at ∞ (a regular function of order 0 rather than order $a > 0$).

Notice that $\ell(x) = \int_1^x \ell_0(t)dt/t$ for some function ℓ_0 slowly varying at ∞ if and only if the function $x\ell'(x)$ is slowly varying at ∞; when ℓ is of the canonical form $\ell(x) = \exp(\int_1^x \epsilon(t)dt/t)$ with $\epsilon(t) \to 0$, this is exactly the case where $\epsilon(.)$ is slowly varying at ∞.

Corollary 1.1 is a direct consequence of Theorem 1.2. Notice that $x\ell'(x) = \ell_0(x)$. Let $\psi(x) = \inf\{\ell'(t) : 1 \le t \le x\}$, for $x \ge 1$. Since $\ell'(x)$ is a regularly varying function of order -1 ($x\ell'(x)$ is slowly varying), we have $\ell'(x) \sim \psi(x)$ when $x \to \infty$ (see [11, Theorem 1.5.3]), where ψ is positive and nonincreasing; so $\ell(x) \asymp \ell_1(x) := \int_1^x \psi(t)dt$ $(x \geq 1)$, and ℓ_1 is a positive and concave function on $[1, \infty)$. Here, as usual, we write

$$f(x) \asymp g(x) \qquad \text{if} \quad 0 < \liminf_{x\to\infty} \frac{f(x)}{g(x)} \leq \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty\,, \tag{1.7}$$

and $f(x) \sim g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 1$. Therefore we can apply Theorem 1.2 to ℓ_1 to obtain the conclusion of the corollary.

By the same method, we can consider some slightly different classes of functions. For example, we can show the following result similar to Theorem 1.1 of Alsmeyer and Rösler (2004) where the Galton-Watson case was considered.

Corollary 1.2 *Let ϕ be positive and convex on $[0, \infty)$ with positive concave derivative ϕ' on $(0, \infty)$. Define*

$$\tilde{\phi}(x) = \begin{cases} \int_1^x \frac{\phi'(t)}{t} dt & \text{if } x > 1; \\ 0 & \text{if } x \leq 1. \end{cases}$$

If $\mathbb{E}m_0^{-1} < 1$ and $\mathbb{E}W_1\tilde{\phi}(W_1) < \infty$, then

$$\mathbb{E}\phi(W^*) < \infty \qquad \text{and} \qquad \mathbb{E}\phi(W) < \infty\,.$$

Notice that Theorem 1.2 also gives an information about the integrability of W^* and thus the non-degeneration of W. In fact, taking $\ell(x) \equiv 1$ in Theorem 1.2, we see that if $\mathbb{E}W_1 \ln^+ W_1 < \infty$ and $\mathbb{E}m_0^{-\delta_0} < \infty$ for some $\delta_0 > 0$, then $\mathbb{E}W^* < \infty$. Here and hereafter, we write $\ln^+ x = \max(0, \ln x)$ and $\ln^- x = \max(0, -\ln x)$. Actually, by the argument in the proof of Theorem 1.2, the above moment condition on m_0 can be relaxed, as shown in the following theorem.

Theorem 1.3 *Assume that $\mathbb{E}(\ln^- m_0)^2 < \infty$. If $\mathbb{E}W_1 \ln^+ W_1 < \infty$, then $\mathbb{E}W^* < \infty$.*

Notice that $\mathbb{E}W^* < \infty$ implies $EW = 1$ by the dominated convergence theorem. Therefore Theorem 1.3 implies the classical theorem (the sufficiency) of Kesten-Stigum (1966) on the Galton-Watson process. It gives a new proof of the corresponding result of Athreya and Karlin (1971b) (see also Tanny (1988)) for a branching process in a random environment, under the extra condition that $\mathbb{E}(\ln^- m_0)^2 < \infty$. (Notice that the supercritical condition $\mathbb{E}\ln m_0 > 0$ implies $\mathbb{E}\ln^- m_0 < \infty$.) Although we need this extra condition, the conclusion that $\mathbb{E}W^* < \infty$ may be useful in applications; we do not know whether this conclusion is equivalent to $EW = 1$. (It is known (see [25]) that the condition $\mathbb{E}W_1 \ln^+ W_1 < \infty$ is equivalent to $EW = 1$; in the Galton-Watson case, it is also known that this condition is also equivalent to $\mathbb{E}W^* < \infty$. But we do not know whether the same conclusion remains true for the random environment case; Theorem 1.3 shows that this is indeed the case if $\mathbb{E}(\ln^- m_0)^2 < \infty$.)

In the Galton-Watson case, Alsmeyer and Rösler (2004) used a similar argument (also based on convex inequalities for martingales) to show the non-degeneration of W. But our approach is more direct, as we do not use their Lemma 4.5.

The rest of the paper is organized as follows. In Section 2, we establish key inequalities based on convex inequalities for martingales. In Section 3, we give smoothed versions of regularly varying functions in order to use the key inequalities of Section 2. Theorem 1.1 is proved in Section 4, while Theorems 1.2 and 1.3 are proved in Sections 5 and 6, respectively.

In enclosing the introduction, we mention that the argument of this paper can be adapted to weighted branching processes, thus enabling us to improve the results of Bingham and Doney (1975) for Crump-Mode and Jirina processes, those of Alsmeyer and Kuhlbusch (2009) for branching random walks, and to extend their results to the random environment case (including the weighted branching processes considered by Kuhlbusch (2004)). This will be done in the forthcoming paper [24].

2 Key Inequalities

In this section, we show key inequalities that will be used for the proof of main theorems. As in Alsmeyer and Rösler (2004), our argument is based on convex inequalities for martingales.

We first introduce some notations. For a finite sequence $u \in \bigcup_{n=0}^{\infty} \mathbb{N}^{*n}$ ($\mathbb{N}^{*0} = \{\emptyset\}$ by convention), set $\tilde{X}_u = \frac{X_u}{m_{|u|}} - 1$. For $n \geq 1$, write

$$D_n = W_n - W_{n-1} = \frac{1}{\Pi_{n-1}} \sum_{|u|=n-1} \tilde{X}_u. \tag{2.1}$$

Then $W^* = \sup_{n\geq 0} W_n$ can be written as

$$W^* = 1 + \sup_{n\geq 1}(D_1 + \ldots + D_n).$$

For convenience, let $\tilde{X}_n = \tilde{X}_{1_n}$ where 1_n denotes the sequence of length n whose components are all equal to 1, with the convention that $1_0 = \emptyset$. Thus when u is a sequence of length n, $\tilde{X}_u$ has the same distribution as $\tilde{X}_n$ under $\mathbb{P}_\zeta$ and $\mathbb{P}$. Define

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \qquad \text{and} \qquad \mathcal{F}_n = \sigma\{\zeta_k, \tilde{X}_u : k < n, |u| < n\} \quad \text{for } n \geq 1. \tag{2.2}$$

Then $(W_n, \mathcal{F}_n)_{n\geq 0}$ also forms a nonnegative martingale under $\mathbb{P}$, as

$$\mathbb{E}(W_n|\mathcal{F}_{n-1}) = \mathbb{E}(\mathbb{E}(W_n|\mathcal{E}_{n-1})|\mathcal{F}_{n-1}) = \mathbb{E}(W_{n-1}|\mathcal{F}_{n-1}) = W_{n-1}.$$

For technical reasons, we will use the martingale $(W_n, \mathcal{F}_n)$, rather than the more frequently used one $(W_n, \mathcal{E}_n)$. We will explain this after the proof of Theorem 1.1. For convenience, we shall write for $n \geq 0$,

$$\mathbb{P}_n(\cdot) = \mathbb{P}(\cdot|\mathcal{F}_n) \qquad \text{and} \qquad \mathbb{E}_n(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_n). \tag{2.3}$$

The terms "increasing" and "decreasing" will be used in the wide sense.

Theorem 2.1 *Let ϕ be convex and increasing with $\phi(0) = 0$ and $\phi(2x) \leq c\phi(x)$ for some constant $c \in (0, \infty)$ and all $x > 0$. Let $\beta \in (1, 2]$.*

(i) If the function $x \mapsto \phi(x^{1/\beta})$ is convex and $\mathbb{E}|\tilde{X}_1|^\beta < \infty$, then writing $A = \sum_{n=1}^{\infty} \frac{1}{\Pi_{n-1}^{\beta-1}}$ which is a.s. finite, we have

$$\begin{aligned}\mathbb{E}\phi(W^* - 1) \quad &\leq \quad C \sum_{n=1}^{\infty} \Bigg(\mathbb{E}\left(\frac{1}{A\Pi_{n-1}^{\beta-1}} \phi(A^{1/\beta} W_{n-1}^{1/\beta}) \right) \\ &\qquad + \mathbb{E}\phi\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}^{(\beta-1)/\beta}} \cdot W_{n-1}^{1/\beta} \right) \Bigg),\end{aligned} \tag{2.4}$$

where $C > 0$ is a constant depending only on c, β and $\mathbb{E}|\tilde{X}_1|^\beta$.

(ii) If the function $x \mapsto \phi(x^{1/\beta})$ is concave, then

$$\mathbb{E}\phi(W^* - 1) \leq C \sum_{n=1}^{\infty} \mathbb{E}\Pi_{n-1}\phi\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}}\right), \tag{2.5}$$

where $C > 0$ is a constant depending only on c.

For the proof, as in Alsmeyer and Rösler (2004), we shall use the BDG (Burkholder-Davis-Gundy) inequalities. For convenience of readers let us first recall these inequalities and state some consequences that we will use. For a martingale sequence $\{(f_n, \mathcal{G}_n) : n \geq 1\}$ defined on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$, set $f_0 = 0$, $\mathcal{G}_0 = \{\emptyset, \Omega\}$, $d_n = f_n - f_{n-1}$ for $n \geq 1$,

$$f^* = \sup_{n\geq 1} |f_n| \quad \text{and} \quad d^* = \sup_{n\geq 1} |d_n|.$$

Lemma 2.1 *Let Φ be an increasing and continuous real function on $[0, \infty)$, with $\Phi(0) = 0$ and $\Phi(2\lambda) \leq c\Phi(\lambda)$ for some c in $(0, \infty)$ and all $\lambda > 0$.*

(a) For every $\beta \in [1,2]$, there exists a constant $B = B_{c,\beta} \in (0,\infty)$ depending only on c and β such that for any martingale $\{(f_n, \mathcal{G}_n) : n \geq 1\}$,

$$\mathbb{E}\Phi(f^*) \leq B\mathbb{E}\Phi(s(\beta)) + B\mathbb{E}\Phi(d^*), \text{ where } s(\beta) = \left(\sum_{n=1}^{\infty} \mathbb{E}\{|d_n|^\beta | \mathcal{G}_{n-1}\}\right)^{1/\beta},$$

and

$$\mathbb{E}\Phi(f^*) \leq B\mathbb{E}\Phi(s(\beta)) + B\sum_{n=1}^{\infty} \mathbb{E}\Phi(|d_n|).$$

(b) If Φ is convex on $[0,\infty)$, then there exist some constants $A = A_c \in (0,\infty)$ and $B = B_c \in (0,\infty)$ depending only on c such that for any martingale $\{(f_n, \mathcal{G}_n) : n \geq 1\}$,

$$A\mathbb{E}\Phi(S) \leq \mathbb{E}\Phi(f^*) \leq B\mathbb{E}\Phi(S), \text{ where } S = \left(\sum_{n=1}^{\infty} d_n^2\right)^{1/2}.$$

Moreover, for any $\beta \in (0,2]$,

$$\mathbb{E}\Phi(f^*) \leq B\mathbb{E}\Phi(S(\beta)), \text{ where } S(\beta) = \left(\sum_{n=1}^{\infty} |d_n|^\beta\right)^{1/\beta}.$$

If additionally, for some $\beta \in (0,2]$ the function $\Phi_{1/\beta}(x) := \Phi(x^{1/\beta})$ is sub-additive on $[0,\infty)$, then

$$\mathbb{E}\Phi(f^*) \leq B\mathbb{E}\sum_{n=1}^{\infty} \Phi(|d_n|).$$

In fact, in Part (a), the first inequality is the general form of the BDG inequality (cf. [12], Chap. 11, p.427, Theorem 2); the second follows from the first because $\Phi(d^*) \leq \sum_{n=1}^{\infty} \Phi(|d_n|)$. In Part (b), the first inequality is the usual form of the BDG inequality (cf. [12], Chap. 11, p.425, Theorem 1); the second follows from the first because $\left(\sum_{n=1}^{\infty} d_n^2\right)^{1/2} \leq \left(\sum_{n=1}^{\infty} |d_n|^\beta\right)^{1/\beta}$ for any $\beta \in (0,2]$ (this can be seen by the sub-additivity of the function $x \mapsto x^{\beta/2}$); the third follows from the second because

$$\Phi(S(\beta)) = \Phi_{1/\beta}(\sum_{n=1}^{\infty} |d_n|^\beta) \leq \sum_{n=1}^{\infty} \Phi_{1/\beta}(|d_n|^\beta) = \sum_{n=1}^{\infty} \Phi(|d_n|).$$

The second inequality of Part (b) may be considered as a counter part of the first inequality of Part (a). Assuming the concavity of the derivative of Φ instead of the subadditivity of $\Phi_{1/\beta}$, Topchii and Vatutin (cf. [26], Theorem 2) also established the third inequality of Part (b).

Proof of Theorem 2.1 (i) We first notice that by the law of large numbers, $\lim_n (\frac{1}{\Pi_n})^{1/n} = e^{-\mathbb{E}\ln m_0} < 1$ a.s., so that $A < \infty$ a.s. (for $\beta > 1$). We now begin with the proof of inequality (2.4). By Lemma 2.1 (a),

$$\mathbb{E}\phi(W^* - 1) \leq B\left(\mathbb{E}\phi\left(\left(\sum_{n=1}^{\infty} \mathbb{E}_{n-1}|D_n|^\beta\right)^{\frac{1}{\beta}}\right) + \sum_{n=1}^{\infty} \mathbb{E}\phi(|D_n|)\right), \tag{2.6}$$

where $B > 0$ is a constant depending only on c and β.

Let $\tilde{X}(1), \ldots, \tilde{X}(Z_{n-1})$ be an enumeration of $\{\tilde{X}_u : u \in T_{n-1}\}$. By the fact that $\mathbb{E}_\zeta \tilde{X}(k) = 0$ and the independence of $\{\tilde{X}_u\}$ under $\mathbb{P}_\zeta$, it can be easily seen that, under $\mathbb{P}_{n-1}$, $\{\tilde{X}(1), \ldots, \tilde{X}(Z_{n-1})\}$ is a sequence of martingale differences with respect to the natural filtration

$$\tilde{\mathcal{F}}_k = \sigma\{\zeta_l, X_u : l < n-1, |u| < n-1, \tilde{X}(1), \ldots, \tilde{X}(k)\}, \qquad k \geq 1. \tag{2.7}$$

To this martingale difference sequence, using Lemma 2.1 (b) (the third inequality of Part (b) applied to $\Phi(x) = x^\beta$, noting that $\Phi_{1/2}(x) = x^{\beta/2}$ is sub-additive), we obtain

$$
\begin{aligned}
\mathbb{E}_{n-1}|D_n|^\beta &= \mathbb{E}_{n-1}\left|\frac{\sum_{|u|=n-1}\tilde{X}_u}{\Pi_{n-1}}\right|^\beta \\
&\leq B\mathbb{E}_{n-1}\sum_{|u|=n-1}\frac{|\tilde{X}_u|^\beta}{\Pi_{n-1}^\beta} \\
&= B\frac{Z_{n-1}}{\Pi_{n-1}^\beta}\cdot\mathbb{E}_{n-1}|\tilde{X}_{n-1}|^\beta \\
&= C\frac{W_{n-1}}{\Pi_{n-1}^{\beta-1}},
\end{aligned}
\tag{2.8}
$$

where $C = B\mathbb{E}|\tilde{X}_1|^\beta < \infty$. (This can also be shown by the usual Marcinkiewicz-Zygmund inequality first conditional on $\mathcal{E}_n$, or by Topchii and Vatutin's inequality ([26], Theorem 2).) Since $\phi_{1/\beta}(x) := \phi(x^{1/\beta})$ is convex and $\sum_{n=1}^\infty \frac{1}{A\Pi_{n-1}^{\beta-1}} = 1$, it follows that

$$
\begin{aligned}
\mathbb{E}\phi\left(\left(\sum_{n=1}^\infty \mathbb{E}_{n-1}|D_n|^\beta\right)^{\frac{1}{\beta}}\right) &= \mathbb{E}\phi_{1/\beta}\Big(\sum_{n=1}^\infty \mathbb{E}_{n-1}|D_n|^\beta\Big) \\
&\leq \mathbb{E}\phi_{1/\beta}\Big(\sum_{n=1}^\infty \frac{1}{A\Pi_{n-1}^{\beta-1}}\cdot ACW_{n-1}\Big) \\
&\leq \mathbb{E}\sum_{n=1}^\infty \frac{1}{A\Pi_{n-1}^{\beta-1}}\phi_{1/\beta}(ACW_{n-1}) \\
&= \mathbb{E}\sum_{n=1}^\infty \frac{1}{A\Pi_{n-1}^{\beta-1}}\cdot\phi\left(C^{1/\beta}A^{1/\beta}W_{n-1}^{1/\beta}\right) \\
&\leq C_1\mathbb{E}\sum_{n=1}^\infty \frac{1}{A\Pi_{n-1}^{\beta-1}}\cdot\phi\left(A^{1/\beta}W_{n-1}^{1/\beta}\right),
\end{aligned}
\tag{2.9}
$$

where $C_1 > 0$ is a constant depending only on C and c. For the second part of (2.6), by Lemma 2.1 (b) and the convexity of $\phi_{1/\beta}(x) = \phi(x^{1/\beta})$, we have (using the fact that $\sum_{|u|=n-1}\frac{1}{Z_{n-1}} = 1$):

$$
\begin{aligned}
\mathbb{E}_{n-1}\phi(|D_n|) &\leq B\mathbb{E}_{n-1}\phi_{1/\beta}\left(\sum_{|u|=n-1}\frac{|\tilde{X}_u|^\beta}{\Pi_{n-1}^\beta}\right) \\
&= B\mathbb{E}_{n-1}\phi_{1/\beta}\left(\sum_{|u|=n-1}\frac{1}{Z_{n-1}}\frac{Z_{n-1}\,|\tilde{X}_u|^\beta}{\Pi_{n-1}^\beta}\right) \\
&\leq B\mathbb{E}_{n-1}\sum_{|u|=n-1}\frac{1}{Z_{n-1}}\phi_{1/\beta}\Big(\frac{Z_{n-1}|\tilde{X}_u|^\beta}{\Pi_{n-1}^\beta}\Big) \\
&= B\mathbb{E}_{n-1}\sum_{|u|=n-1}\frac{1}{Z_{n-1}}\phi\left(\frac{|\tilde{X}_u|}{\Pi_{n-1}}\cdot Z_{n-1}^{1/\beta}\right) \\
&= B\mathbb{E}_{n-1}\phi\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}^{(\beta-1)/\beta}}\cdot W_{n-1}^{1/\beta}\right).
\end{aligned}
\tag{2.10}
$$

Therefore

$$
\mathbb{E}\phi(|D_n|) \leq B\mathbb{E}\phi\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}^{(\beta-1)/\beta}}W_{n-1}^{1/\beta}\right). \tag{2.11}
$$

(ii) Notice that the concavity of $\phi_{1/\beta}(x) = \phi(x^{1/\beta})$ implies its subadditivity. Therefore by the third inequality of Part (b) of Lemma 2.1,

$$\mathbb{E}\phi(W^* - 1) \quad \leq \quad B \sum_{n \geq 1} \mathbb{E}\phi(|D_n|), \tag{2.12}$$

where $B > 0$ is a constant depending only on c. By the same reason,

$$\begin{aligned} \mathbb{E}_{n-1}\phi(|D_n|) &= \mathbb{E}_{n-1}\phi\left(\left|\frac{1}{\Pi_{n-1}} \sum_{|u|=n-1} \tilde{X}_u\right|\right) \\ &\leq B\mathbb{E}_{n-1} \sum_{|u|=n-1} \phi\left(\frac{|\tilde{X}_u|}{\Pi_{n-1}}\right) \\ &= BZ_{n-1}\mathbb{E}_\zeta \phi\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}}\right). \end{aligned} \tag{2.13}$$

Hence

$$\mathbb{E}\phi(|D_n|) \leq B\mathbb{E}[Z_{n-1}\mathbb{E}_\zeta \phi\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}}\right)] = \mathbb{E}\Pi_{n-1}\phi\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}}\right). \tag{2.14}$$

Combining (2.12) and (2.14), we get (2.5).

3 Smoothed versions of regularly varying functions

In this section, for a regularly varying function, we find some smoothed versions with nice properties that we will need in order to use convex inequalities for martingales.

Lemma 3.1 *Let $\phi(x) = x^\alpha \ell(x)$, with $\alpha > 1$, and $\ell(x) = \exp\left(\int_{a_0}^x \epsilon(u)du/u\right)$ $(x \geq a_0 \geq 0)$ with $\epsilon(x) \to 0$ $(x \to 0)$. Then for each $\beta \in (1, 2]$ with $\beta < \alpha$, there is a function $\phi_1(x) \geq 0$ such that:*

(i) $\phi_1(x) \sim \phi(x)$;

(ii) $\phi_1(x)$ and $\phi_1(x^{1/\beta})$ are convex on $[0, \infty)$;

(iii) $\phi_1(x) = x^\alpha \ell_1(x)$, where $\ell_1(x)$ is slowly varying at ∞ and $\ell_1(x) > 0$ $\forall x \geq 0$.

Proof. Fix $\beta \in (1, 2]$ with $\beta < \alpha$. Notice that the derivative

$$\phi'(x) = x^{\alpha-1}\ell(x)(\alpha + \epsilon(x))$$

behaves like $\alpha x^{\alpha-1}\ell(x)$ as $x \to \infty$. It is therefore natural to define

$$\phi_1(x) = \alpha \int_0^x t^{\alpha-1}\ell(t)dt, \qquad x > a, \tag{3.1}$$

where $a \geq \max(1, a_0)$ is large enough such that $\forall\, x > a$, $\alpha - \beta + \epsilon(x) > 0$, so that

$$\frac{d}{dx}(x^{\alpha-1}\ell(x)) = x^{\alpha-2}\ell(x)(\alpha - 1 + \epsilon(x)) > 0 \qquad \forall x > a, \tag{3.2}$$

and

$$\frac{d}{dx}\left(x^{\frac{\alpha}{\beta}-1}\ell(x^{\frac{1}{\beta}})\right) = x^{\frac{\alpha}{\beta}-2}\ell(x^{\frac{1}{\beta}})\left((\frac{\alpha}{\beta} - 1) + \frac{\epsilon(x^{\frac{1}{\beta}})}{\beta}\right) > 0 \quad \forall x > a^\beta. \tag{3.3}$$

Therefore, $\phi_1(x)$ is convex on (a, ∞) as $\phi_1'(x) = \alpha x^{\alpha-1}\ell(x)$ is increasing; and $\phi_1(x^{1/\beta})$ is convex on $[a^\beta, \infty)$ as

$$\frac{d}{dx}\phi_1(x^{1/\beta}) = \phi_1'(x^{1/\beta}) \cdot \frac{1}{\beta}x^{\frac{1}{\beta}-1} = \frac{\alpha}{\beta}x^{\frac{\alpha}{\beta}-1}\ell(x^{1/\beta}) \qquad (x > a^\beta)$$

is also increasing on (a^β, ∞). Define

$$\phi_1(x) = x^\alpha \ell(a), \qquad \forall x \in [0, a]. \tag{3.4}$$

Then

$$\frac{d}{dx}\phi_1(x) = \alpha x^{\alpha-1}\ell(a) \qquad \forall x \in [0, a], \tag{3.5}$$

and

$$\begin{aligned} \frac{d}{dx}\phi_1(x^{1/\beta}) &= \frac{d}{dx}\left(x^{\alpha/\beta}\ell(a)\right) \\ &= \frac{\alpha}{\beta}x^{\frac{\alpha}{\beta}-1}\ell(a) \qquad \forall x \in [0, a^\beta]. \end{aligned} \tag{3.6}$$

It follows that both $\frac{d}{dx}\phi_1(x)$ and $\frac{d}{dx}\phi_1(x^{1/\beta})$ are increasing on $[0, \infty)$. Therefore both $\phi_1(x)$ and $\phi_1(x^{1/\beta})$ are convex on $[0, \infty)$. Moreover,

$$\lim_{x\to\infty}\frac{\phi_1(x)}{\phi(x)} = \lim_{x\to\infty}\frac{\phi_1'(x)}{\phi'(x)} = 1, \tag{3.7}$$

so that $\phi_1(x) = x^\alpha \ell_1(x)$ for some slowly varying function ℓ_1. If $x > a$, then $\ell_1(x) > 0$ as $\phi_1(x) > 0$; if $x \leq a$, then $\ell_1(x) = \ell(a) > 0$. Therefore, $\ell_1(x) > 0$ $\forall x \geq 0$.

□

Lemma 3.2 *Let ℓ be a positive and increasing function on $[0, \infty)$, concave on (a_0, ∞) for some $a_0 \geq 0$. Then there is a convex increasing function $\phi_1(x) \geq 0$ such that:*

(i) $\phi_1(x) \asymp x\ell(x)$*;*

(ii) $\phi_1(2x) \leq c\phi_1(x)$ *for some constant* $c \in (0, \infty)$ *and all* $x > 0$*;*

(iii) $\phi_1(x^{1/2})$ *is concave on* $(0, \infty)$*.*

Proof. Let

$$\ell_1(x) = \begin{cases} \ell'(a)x & \text{if} \quad x \in [0, a], \\ \ell(x) + c_0 & \text{if} \quad x \in (a, \infty), \end{cases} \tag{3.8}$$

where $a > a_0 > 0$, $c_0 = \ell'(a)a - \ell(a)$, and $\phi_1(x) = \int_0^x \ell_1(t)dt$. We will show that ϕ_1 satisfies the stated properties.

First, ϕ_1 is convex as $\phi_1'(x) = \ell_1(x)$ is increasing on $[0, \infty)$; ϕ_1 is increasing as ℓ_1 is positive on $[0, \infty)$.

Next, for $x > 2a$, as ℓ is increasing, we have $\ell_1(t) \geq \ell_1(\frac{x}{2}) = \ell(\frac{x}{2}) + c_0$ if $t \in [\frac{x}{2}, x]$, and $\ell_1(t) \leq \ell_1(x)$ if $t \in [0, x]$; therefore

$$\frac{x}{2}\left(\ell(\frac{x}{2}) + c_0\right) \leq \phi_1(x) \leq x(\ell(x) + c_0). \tag{3.9}$$

By the concavity of ℓ, for all $x > a$,

$$\begin{aligned} \ell(2x) &= \ell(x) + 2\int_{\frac{x}{2}}^{x} \ell'(2s)ds \\ &\leq \ell(x) + 2\int_0^x \ell'(s)ds \\ &\leq 3\ell(x). \end{aligned} \tag{3.10}$$

(3.9) and (3.10) imply that $\phi_1(x) \asymp x\ell(x)$ and that there is a constant $c \in (0, \infty)$ such that $\ell_1(2x) \leq c\ell_1(x)$ for all $x > 0$.

Moreover, we can prove that $\phi_1(x^{1/2})$ is concave. In fact,

$$\begin{aligned} \frac{d}{dx}\phi_1(x^{1/2}) &= \phi_1'(x^{\frac{1}{2}}) \cdot \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2}\ell_1(x^{\frac{1}{2}}) \cdot x^{-\frac{1}{2}}. \end{aligned} \tag{3.11}$$

Notice that $\frac{\ell_1(t)}{t}$ is decreasing as ℓ_1 is concave with $\ell_1(0) = 0$; hence $\frac{d}{dx}\phi_1(x^{1/2})$ is decreasing, so that $\phi_1(x^{1/2})$ is concave. □

4 Proof of Theorem 1.1

In this section, the letter C will denote a finite and positive constant whose value is not important and may differ from line to line.

Proof of Theorem 1.1. Let $\beta \in (1,2]$ with $\beta < \alpha$. Write $\phi(x) = x^\alpha \ell(x)$. By Lemma 3.1, we can assume that the functions $\phi(x)$ and $\phi(x^{1/\beta})$ are convex on $[0,\infty)$, and $\ell(x) > 0 \ \forall\, x \geq 0$.

(i) We first show that (a) implies (b). By Theorem 2.1(i), we obtain

$$\begin{aligned} \mathbb{E}\phi(W^* - 1) \quad &\leq \quad C \sum_{n=1}^{\infty} \left(\mathbb{E}\left(\frac{1}{A\Pi_{n-1}^{\beta-1}} \phi\left(A^{1/\beta} W_{n-1}^{1/\beta} \right) \right) \right. \\ &\qquad \left. + \mathbb{E}\phi\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}^{(\beta-1)/\beta}} \cdot W_{n-1}^{1/\beta} \right) \right) . \end{aligned} \tag{4.1}$$

Notice that $\ell > 0$ on any compact subset of $[0,\infty)$, so by Potter's Theorem (see [11]), for $\delta > 0$ which will be determined later, there exists $C = C(\ell,\delta) > 1$ such that $\ell(x) \leq C \max(x^\delta, x^{-\delta})$ for all $x > 0$. Hence for the first part of (4.1), we have

$$\begin{aligned} \mathbb{E}\left(\frac{1}{A\Pi_{n-1}^{\beta-1}} \phi\left(A^{1/\beta} W_{n-1}^{1/\beta} \right) \right) \quad &= \quad \mathbb{E}\left(\Pi_{n-1}^{1-\beta} A^{\frac{\alpha}{\beta}-1} W_{n-1}^{\frac{\alpha}{\beta}} \ell\left(A^{1/\beta} W_{n-1}^{1/\beta} \right) \right) \\ &\leq \quad C(I_1^+(n) + I_1^-(n)), \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} I_1^+(n) &= \mathbb{E}\Pi_{n-1}^{1-\beta} A^{\frac{\alpha+\delta}{\beta}-1} W_{n-1}^{\frac{\alpha+\delta}{\beta}}, \\ I_1^-(n) &= \mathbb{E}\Pi_{n-1}^{1-\beta} A^{\frac{\alpha-\delta}{\beta}-1} W_{n-1}^{\frac{\alpha-\delta}{\beta}}. \end{aligned}$$

Recall that Z_{n-1} is an integer-valued random variable with $\mathbb{E}_\zeta Z_{n-1} = \Pi_{n-1}$. Choose $\delta > 0$ small enough such that $\beta - 1 - 2\delta > 0$. Then by Hölder's inequality, we obtain

$$\begin{aligned} \mathbb{E}_\zeta Z_{n-1}^{\frac{\alpha+\delta}{\beta}} \quad &= \quad \mathbb{E}_\zeta \left(Z_{n-1}^{\frac{\alpha+\delta-(\beta-1)}{\beta}} \cdot Z_{n-1}^{\frac{\beta-1}{\beta}} \right) \\ &\leq \quad (\mathbb{E}_\zeta Z_{n-1}^{\alpha+\delta-(\beta-1)})^{\frac{1}{\beta}} \cdot (\mathbb{E}_\zeta Z_{n-1})^{\frac{\beta-1}{\beta}} \\ &\leq \quad \left(\mathbb{E}_\zeta Z_{n-1}^{\alpha+\delta-(\beta-1)+(\beta-1-2\delta)} \right)^{\frac{1}{\beta}} \cdot (\mathbb{E}_\zeta Z_{n-1})^{\frac{\beta-1}{\beta}} \\ &= \quad \Pi_{n-1}^{\frac{\alpha+\beta-1-\delta}{\beta}} \left(\mathbb{E}_\zeta W_{n-1}^{\alpha-\delta} \right)^{\frac{1}{\beta}} . \end{aligned} \tag{4.3}$$

Therefore,

$$\begin{aligned} I_1^+(n) \quad &= \quad \mathbb{E}\left(\Pi_{n-1}^{1-\beta} A^{\frac{\alpha+\delta}{\beta}-1} \cdot \frac{Z_{n-1}^{\frac{\alpha+\delta}{\beta}}}{\Pi_{n-1}^{\frac{\alpha+\delta}{\beta}}} \right) \\ &\leq \quad \mathbb{E}\left(\Pi_{n-1}^{-\frac{(\beta-1)^2+2\delta}{\beta}} A^{\frac{\alpha+\delta}{\beta}-1} \left(\mathbb{E}_\zeta W_{n-1}^{\alpha-\delta} \right)^{\frac{1}{\beta}} \right) . \end{aligned} \tag{4.4}$$

Using Hölder's inequality twice, we see that

$$\begin{aligned} I_1^+(n) \quad &\leq \quad \left(\mathbb{E}W_{n-1}^{\alpha-\delta} \right)^{\frac{1}{\beta}} \cdot \left(\mathbb{E}\Pi_{n-1}^{-\frac{(\beta-1)^2+2\delta}{\beta-1}} A^{\frac{\alpha+\delta-\beta}{\beta-1}} \right)^{\frac{\beta-1}{\beta}} \\ &\leq \quad \left(\mathbb{E}W_{n-1}^{\alpha-\delta} \right)^{\frac{1}{\beta}} \cdot \left(\mathbb{E}\Pi_{n-1}^{-\frac{(\beta-1)^2+2\delta}{\beta-1}p} \right)^{\frac{\beta-1}{p\beta}} \cdot \left(\mathbb{E}A^{\frac{\alpha+\delta-\beta}{\beta-1}p^*} \right)^{\frac{\beta-1}{p^*\beta}} , \end{aligned} \tag{4.5}$$

where $p > 1$, $p^* > 1$ and $\frac{1}{p} + \frac{1}{p^*} = 1$. By Potter's Theorem, there exists $C = C(\ell,\delta) > 0$ such that $\ell(x) \geq C x^{-\delta}$ for all $x \geq 1$. This yields

$$\mathbb{E}|\tilde{X}|^{\alpha-\delta} \leq C(1 + \mathbb{E}\phi(|\tilde{X}|)) < \infty . \tag{4.6}$$

Since $\alpha \in \mathrm{Int}\{a > 1 : \mathbb{E}m_0^{1-a} < 1\}$, there exists $\delta_0 \in (0,1)$ such that

$$\mathbb{E}m_0^{1-(\alpha+\delta_0)} < 1.$$

Notice that the function $\rho(x) = \mathbb{E}m_0^{1-x}$ is convex with $\rho(1) = 1$, so $\rho(\alpha+\delta_0) < 1$ implies $\rho(x) < 1$ for all $1 < x < \alpha + \delta_0$; in particular, $\rho(\alpha - \delta) < 1$. Hence, by Lemma 1.1,

$$\sup_{n\geq 1} \mathbb{E}W_{n-1}^{\alpha-\delta} < \infty\,. \tag{4.7}$$

We choose $p = 1 + \frac{(\alpha+\delta-\beta)(\beta-1)}{(\beta-1)^2+2\delta}$ so that $p_1 := \frac{\alpha+\delta-\beta}{\beta-1}p^* = \frac{(\beta-1)^2+2\delta}{(\beta-1)^2}p$. As $p_1(\beta-1) \in (1, \alpha+\delta_0)$ when δ is small enough, we get $\mathbb{E}\Pi_{n-1}^{-p_1(\beta-1)} = a^{n-1}$ with $a = \mathbb{E}m_0^{-p_1(\beta-1)} < 1$; moreover, by the triangular inequality for the norm $\|\cdot\|_{p_1}$ in L^{p_1},

$$\|A\|_{p_1} \leq \sum_{n=1}^{\infty} \|\Pi_{n-1}^{-(\beta-1)}\|_{p_1} = \sum_{n=1}^{\infty} a^{(n-1)/p_1} < \infty\,.$$

Therefore,

$$\sum_{n=1}^{\infty} I_1^+(n) < \infty\,. \tag{4.8}$$

We use a similar argument to estimate $I_1^-(n)$. This time, instead of (4.4), we have

$$I_1^-(n) \leq \mathbb{E}\left(\Pi_{n-1}^{1-\beta} A^{\frac{\alpha-\delta}{\beta}-1} (\mathbb{E}_\zeta W_{n-1}^{\alpha-\delta})^{\frac{1}{\beta}}\right). \tag{4.9}$$

Proceeding in the same way as before, we obtain

$$\sum_{n=1}^{\infty} I_1^-(n) < \infty\,. \tag{4.10}$$

Hence

$$\sum_{n=1}^{\infty} \mathbb{E}\left(\frac{1}{A\Pi_{n-1}^{\beta-1}}\phi\left(A^{1/\beta}W_{n-1}^{1/\beta}\right)\right) < \infty\,. \tag{4.11}$$

We now consider the second part of (4.1). Again by Potter's theorem and the fact that $\tilde{X}_{n-1}$ is independent of W_{n-1} and Π_{n-1} (under $\mathbb{P}$), we have

$$\begin{aligned}
\mathbb{E}\phi\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}^{(\beta-1)/\beta}} \cdot W_{n-1}^{1/\beta}\right) &= \mathbb{E}\left(\frac{W_{n-1}}{\Pi_{n-1}^{\beta-1}}\right)^{\frac{\alpha}{\beta}} |\tilde{X}_{n-1}|^\alpha \ell\left(\left(\frac{W_{n-1}}{\Pi_{n-1}^{\beta-1}}\right)^{\frac{1}{\beta}} |\tilde{X}_{n-1}|\right) \\
&\leq C\mathbb{E}\phi(|\tilde{X}_0|) \cdot (I_2^+(n) + I_2^-(n)),
\end{aligned} \tag{4.12}$$

where

$$I_2^+(n) = \mathbb{E}\left(\frac{W_{n-1}}{\Pi_{n-1}^{\beta-1}}\right)^{\frac{\alpha+\delta}{\beta}},$$

$$I_2^-(n) = \mathbb{E}\left(\frac{W_{n-1}}{\Pi_{n-1}^{\beta-1}}\right)^{\frac{\alpha-\delta}{\beta}},$$

$\delta \leq \delta_0$, $C = C(\ell_1, \delta, \beta) > 0$ is a constant depending only on ℓ_1, δ and β. Here we have used the fact that under $\mathbb{P}$, each $\tilde{X}_{n-1}$ has the same distribution as $\tilde{X}_0$. We can estimate $I_2^+(n)$ as we have done for $I_1^+(n)$:

we have

$$\begin{aligned}
I_2^+(n) &= \mathbb{E}\left(\Pi_{n-1}^{-(\alpha+\delta)}\cdot\mathbb{E}_\zeta Z_{n-1}^{\frac{\alpha+\delta}{\beta}}\right)\\
&\le \mathbb{E}\left(\Pi_{n-1}^{-(\alpha+\delta-\frac{\beta-1}{\beta})}\cdot\left(\mathbb{E}_\zeta Z_{n-1}^{\alpha+\delta-(\beta-1)}\right)^{\frac{1}{\beta}}(\mathbb{E}_\zeta W_{n-1})^{\frac{\beta-1}{\beta}}\right)\\
&= \mathbb{E}\left(\Pi_{n-1}^{-(\alpha+\delta-\frac{\beta-1}{\beta})}\cdot\left(\mathbb{E}_\zeta Z_{n-1}^{\alpha+\delta-(\beta-1)}\right)^{\frac{1}{\beta}}\right)\\
&\le \mathbb{E}\left(\Pi_{n-1}^{-(\alpha+\delta-\frac{\beta-1}{\beta})}\left(\mathbb{E}_\zeta Z_{n-1}^{\alpha+\delta-(\beta-1)+(\beta-1-2\delta)}\right)^{\frac{1}{\beta}}\right)\\
&= \mathbb{E}\left(\Pi_{n-1}^{-\frac{(\alpha+\delta-1)(\beta-1)}{\beta}}\cdot\left(\mathbb{E}_\zeta W_{n-1}^{\alpha-\delta}\right)^{\frac{1}{\beta}}\right)\\
&\le \left(\mathbb{E}W_{n-1}^{\alpha-\delta}\right)^{\frac{1}{\beta}}\left(\mathbb{E}\Pi_{n-1}^{1-(\alpha+\delta)}\right)^{\frac{\beta-1}{\beta}}. \qquad (4.13)
\end{aligned}$$

It follows that

$$\sum_{n\ge1}^{\infty} I_2^+(n) \le \left(\sup_{n\ge1}\mathbb{E}W_{n-1}^{\alpha-\delta}\right)^{\frac{1}{\beta}}\cdot\left(\sum_{n=1}^{\infty}\left(\mathbb{E}m_0^{1-(\alpha+\delta)}\right)^{\frac{n(\beta-1)}{\beta}}\right)<\infty. \qquad (4.14)$$

Similarly we obtain

$$I_2^-(n) \le (\mathbb{E}W_{n-1}^{\alpha-\delta})^{\frac{1}{\beta}}\left(\mathbb{E}\Pi_{n-1}^{1-(\alpha-\delta)}\right)^{\frac{\beta-1}{\beta}} \qquad (4.15)$$

and

$$\sum_{n\ge1}^{\infty} I_2^-(n) \le \left(\sup_{n\ge1}\mathbb{E}W_{n-1}^{\alpha-\delta}\right)^{\frac{1}{\beta}}\cdot\left(\sum_{n=1}^{\infty}\left(\mathbb{E}m_0^{1-(\alpha-\delta)}\right)^{\frac{n(\beta-1)}{\beta}}\right)<\infty. \qquad (4.16)$$

Therefore,

$$\sum_{n=1}^{\infty}\mathbb{E}\phi\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}^{(\beta-1)/\beta}}W_{n-1}^{1/\beta}\right)<\infty. \qquad (4.17)$$

Combining (4.1), (4.11) and (4.17), we get

$$\mathbb{E}\phi(W^*-1)<\infty, \qquad (4.18)$$

which is equivalent to $\mathbb{E}\phi(W^*)<\infty$.

(ii) We next show that (b) implies (c). Obviously,

$$\mathbb{E}\phi(W)\le\mathbb{E}\phi(W^*)<\infty\,;$$

by Jensen's inequality, for any $n\ge1$,

$$\mathbb{E}\phi(W_n)\ge\phi(\mathbb{E}W_n)=\phi(1)>0.$$

So by the dominated convergence theorem, we see that

$$\mathbb{E}\phi(W)=\lim_{n\to\infty}\mathbb{E}\phi(W_n)\ge\phi(1)>0.$$

(iii) We finally show that (c) implies (a). Notice that the limit W satisfies the equation

$$W=\sum_{i=1}^{Z_1}\frac{W^{(i)}}{m_0}, \qquad (4.19)$$

where under $\mathbb{P}_\zeta$, $(W^{(i)})$ are independent of each other, and have the same law as W under $\mathbb{P}_{T\zeta}$: $\mathbb{P}_\zeta(W^{(i)}\in\cdot)=\mathbb{P}_{T\zeta}(W\in\cdot)$, T being the usual translation: $T\zeta=(\zeta_1,\zeta_2,\ldots)$ if $\zeta=(\zeta_0,\zeta_1,\ldots)$. By Jensen's inequality, writting $\mathbb{E}_{\zeta,1}(\cdot)=\mathbb{E}_\zeta(\cdot|\mathcal{F}_1)$, we have

$$\begin{aligned}
\mathbb{E}_\zeta\phi(W)=\mathbb{E}_\zeta\phi\left(\sum_{i=1}^{Z_1}\frac{W^{(i)}}{m_0}\right) &\ge \mathbb{E}_\zeta\phi\left(\mathbb{E}_{\zeta,1}\sum_{i=1}^{Z_1}\frac{W^{(i)}}{m_0}\right)\\
&= \mathbb{E}_\zeta\phi(Z_1/m_0)=\mathbb{E}_\zeta\phi(W_1). \qquad (4.20)
\end{aligned}$$

Therefore

$$\mathbb{E}\phi(W_1) \leq \mathbb{E}\phi(W). \tag{4.21}$$

$\square$

Remark. For technical reasons, in the proof of Theorem 1.1, we have used the martingale $(W_n, \mathcal{F}_n)$ under $\mathbb{P}$ rather than the more natural martingale $(W_n, \mathcal{E}_n)$ under $\mathbb{P}_\zeta$. In fact, if we take the later martingale, then instead of (2.6), we have

$$\mathbb{E}_\zeta \phi(W^* - 1) \leq B \left(\mathbb{E}_\zeta \phi \left(\left(\sum_{n=1}^{\infty} \mathbb{E}_\zeta (|D_n|^\beta | \mathcal{E}_{n-1}) \right)^{\frac{1}{\beta}} \right) + \sum_{n=1}^{\infty} \mathbb{E}_\zeta \phi(|D_n|) \right); \tag{4.22}$$

instead of (2.8), we obtain

$$\mathbb{E}_\zeta (|D_n|^\beta | \mathcal{E}_{n-1}) \leq B \frac{\sigma_{n-1}^\beta(\beta)}{\Pi_{n-1}^{\beta-1}} W_{n-1}. \tag{4.23}$$

Taking expectations and using the same argument as in part (i) of the proof of Theorem 1.1, we obtain

$$\mathbb{E}\phi(W^* - 1) \leq C \left(\sum_{n=1}^{\infty} (\tilde{I}_1^+(n) + \tilde{I}_1^-(n)) + \sum_{n=1}^{\infty} \mathbb{E}\phi(|D_n|) \right), \tag{4.24}$$

where

$$\tilde{I}_1^+(n) = \mathbb{E}\Pi_{n-1}^{1-\beta} A^{\frac{\alpha+\delta-\beta}{\beta}} [\sigma_{n-1}(\beta)]^{\alpha+\delta} W_{n-1}^{\frac{\alpha+\delta}{\beta}},$$
$$\tilde{I}_1^-(n) = \mathbb{E}\Pi_{n-1}^{1-\beta} A^{\frac{\alpha-\delta-\beta}{\beta}} [\sigma_{n-1}(\beta)]^{\alpha-\delta} W_{n-1}^{\frac{\alpha-\delta}{\beta}}.$$

The problem here is that we have to deal with the extra term $\sigma_{n-1}(\beta)$ in $\tilde{I}_1^\pm(n)$. We can do this by Hölder's inequality, but we then need an extra moment condition on $\sigma_{n-1}(\beta)$. Elementary calculations show that if for some positive number δ_0, either (a) $\alpha < 2$ and $\mathbb{E}[\sigma_0(\alpha)]^{\alpha(\alpha+\delta_0)} < \infty$, or (b) $\alpha \geq 2$ and $\mathbb{E}[\sigma_0(2)]^{2(\alpha+\delta_0)} < \infty$, then $\sum_{n=1}^\infty \tilde{I}_1^\pm(n) < \infty$, provided that $\mathbb{E}W_1^\alpha \ell(W_1) < \infty$. This leads a proof of Theorem 1.1 under the preceding extra moment condition.

5 Proofs of Theorem 1.2 and Corollary 1.2

As in the preceding section, we still use the letter C to denote a finite and positive constant whose value is not important and may differ from line to line.

Proof of Theorem 1.2. Let ϕ be ϕ_1 and ℓ be ℓ_1 defined in Lemma 3.2 (the new function ℓ_1 is still denoted ℓ for simplicity; we can replace ℓ by ℓ_1 as $\hat{\ell}(x) \asymp \hat{\ell}_1(x)$).

Notice that under $\mathbb{P}$, $\tilde{X}_{n-1}$ is independent of Π_{n-1}. As ℓ is concave, we have

$$\begin{aligned} \mathbb{E}\Pi_{n-1}\phi\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}}\right) &\leq C\mathbb{E}|\tilde{X}_{n-1}|\ell\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}}\right) \\ &\leq C\mathbb{E}|\tilde{X}_{n-1}|\ell(b^n|\tilde{X}_{n-1}|) \\ &= C\mathbb{E}|\tilde{X}|\ell(b^n|\tilde{X}|), \end{aligned} \tag{5.1}$$

where $\tilde{X}$ is a random variable having the same distribution as $(\tilde{X}_n)_{n\geq 0}$ and $b = \mathbb{E}m_0^{-1} < 1$. According to the inequality (2.5), we have

$$\begin{aligned}
\mathbb{E}\phi(W^*-1) &\le C\sum_{n=1}^{\infty}\mathbb{E}\Pi_{n-1}\phi\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}}\right)\\
&\le C\sum_{n=1}^{\infty}\mathbb{E}|\tilde{X}|\ell(b^n|\tilde{X}|)\\
&\le C\sum_{n=1}^{\infty}\mathbb{E}|\tilde{X}|\int_{b^n|\tilde{X}|}^{b^{n-1}|\tilde{X}|}\frac{\ell(t)}{t}dt\\
&= C\mathbb{E}|\tilde{X}|\int_0^{|\tilde{X}|}\frac{\ell(t)}{t}dt\\
&\le C\mathbb{E}|\tilde{X}|(1+\hat{\ell}(|\tilde{X}|))<\infty\,. \qquad (5.2)
\end{aligned}$$

This yields $\mathbb{E}W^*\ell(W^*)<\infty$, and

$$\mathbb{E}W\ell(W)\le \mathbb{E}W^*\ell(W^*)<\infty\,.$$

If in addition, ℓ is slowly varying at ∞, then we can use Potter's theorem to replace the Jensen's inequality in (5.1), to relax the assumption $\mathbb{E}m_0^{-1}<1$. Recall that for this ℓ, we have shown that

$$\begin{aligned}
\mathbb{E}\phi(W^*-1) &\le C\sum_{n=1}^{\infty}\mathbb{E}|\tilde{X}|\ell\left(\frac{|\tilde{X}|}{\Pi_{n-1}}\right) \qquad (5.3)\\
&= C\sum_{n=1}^{\infty}(I_3(n)+I_3'(n)), \qquad (5.4)
\end{aligned}$$

where

$$\begin{aligned}
I_3(n) &= E|\tilde{X}|\ell\left(\frac{|\tilde{X}|}{\Pi_{n-1}}\right)\mathbf{1}_{\{\Pi_{n-1}^{-1}\le a^{n-1}\}},\\
I_3'(n) &= E|\tilde{X}|\ell\left(\frac{|\tilde{X}|}{\Pi_{n-1}}\right)\mathbf{1}_{\{\Pi_{n-1}^{-1}> a^{n-1}\}},
\end{aligned}$$

$a\in(0,1)$ will be determined later. By the same argument as above, we get

$$\sum_{n=1}^{\infty}I_3(n)\le C\mathbb{E}|\tilde{X}|(\hat{\ell}(|\tilde{X}|)+1)<\infty\,. \qquad (5.5)$$

We now estimate $I_3'(n)$. For fixed n, we divide it into two parts:

$$\begin{aligned}
I_{3,1}'(n) &= \mathbb{E}|\tilde{X}|\ell\left(\frac{|\tilde{X}|}{\Pi_{n-1}}\right)\mathbf{1}_{\{\Pi_{n-1}^{-1}> a^{n-1}\}}\mathbf{1}_{\{|\tilde{X}|a^{n-1}>1\}};\\
I_{3,2}'(n) &= \mathbb{E}|\tilde{X}|\ell\left(\frac{|\tilde{X}|}{\Pi_{n-1}}\right)\mathbf{1}_{\{\Pi_{n-1}^{-1}> a^{n-1}\}}\mathbf{1}_{\{|\tilde{X}|a^{n-1}\le 1\}}.
\end{aligned}$$

As ℓ is increasing and slowly varying at ∞, by Potter's theorem, we have: for $\delta>0$,

$$\begin{aligned}
I_{3,1}'(n) &\le C\mathbb{E}|\tilde{X}|\ell(|\tilde{X}|a^{n-1})(\Pi_{n-1}a^{n-1})^{-\delta}\\
&\le C\mathbb{E}|\tilde{X}|\ell(|\tilde{X}|)(\Pi_{n-1}a^{n-1})^{-\delta}\\
&= C\mathbb{E}|\tilde{X}|\ell(|\tilde{X}|)\cdot(\mathbb{E}m_0^{-\delta}\cdot a^{-\delta})^n a^{\delta}. \qquad (5.6)
\end{aligned}$$

Let $\rho(x)=\mathbb{E}m_0^{-x}$. Since $\rho(\delta_0)<\infty$ and $\rho(x)$ is convex on $(0,\delta_0)$ with $\rho(0)=1$ and $\rho'(0)=-\mathbb{E}\ln m_0<0$, there exists some $\gamma_0>0$ such that

$$\mathbb{E}m_0^{-x}<1,\qquad \forall x\in(0,\gamma_0).$$

Choose $\delta \in (0, \gamma_0)$, and let $0 < a < 1$ be defined by $\mathbb{E}m_0^{-\delta} = a^{2\delta}$. Notice that $\mathbb{E}|\tilde{X}|\ell(|\tilde{X}|) \leq C\mathbb{E}|\tilde{X}|(\hat{\ell}(|\tilde{X}|) + 1) < \infty$. Therefore,

$$\sum_{n=1}^{\infty} I'_{3,1}(n) \leq C\mathbb{E}|\tilde{X}|\ell(|\tilde{X}|) \cdot \sum_{n=1}^{\infty} a^{\delta(n+1)} < \infty\,. \tag{5.7}$$

Similarly, using Potter's theorem in $I'_{3,2}(n)$, we get

$$\begin{aligned} I'_{3,2}(n) &\leq \mathbb{E}|\tilde{X}|\ell(\Pi_{n-1}^{-1}a^{1-n})\mathbf{1}_{\{\Pi_{n-1}^{-1} > a^{n-1}\}} \\ &\leq C\mathbb{E}|\tilde{X}|\ell(1)(\Pi_{n-1}a^{n-1})^{-\delta} \\ &\leq C\mathbb{E}|\tilde{X}| \cdot (\mathbb{E}m_0^{-\delta}a^{-\delta})^n a^\delta \\ &\leq C\mathbb{E}|\tilde{X}| \cdot a^{\delta(n+1)}. \end{aligned} \tag{5.8}$$

Hence

$$\sum_{n=1}^{\infty} I'_{3,2}(n) \leq C\mathbb{E}|\tilde{X}| \cdot \sum_{n=1}^{\infty} a^{\delta(n-1)} < \infty\,. \tag{5.9}$$

Therefore, we have shown that

$$\mathbb{E}\phi(W^* - 1) < \infty\,, \tag{5.10}$$

which is equivalent to $\mathbb{E}\phi(W^*) < \infty$.

$\square$

Proof of Corollary 1.2. Let

$$\phi_1(x) = \begin{cases} \frac{\phi'(1)}{2}x^2 & \text{if } x \leq 1; \\ \phi(x) + c_0 & \text{if } x > 1 \end{cases} \tag{5.11}$$

where $\phi(1) + c_0 = \frac{\phi'(1)}{2}$. Then it is easily seen that $\phi_1 \asymp \phi, \phi_1(0) = 0, \phi_1'(0+) = 0$ and $\int_0^1 \frac{\phi_1'(t)}{t}dt = \phi'(1) < \infty$. Moreover, ϕ_1 is convex with positive concave derivative ϕ_1' on $(0, \infty)$, so that the function $x \mapsto \phi_1(x^{1/2})$ is concave on $(0, \infty)$. Applying the BDG-inequality and the concavity of $\phi_1(x^{1/2})$ (which implies the subadditivity), we obtain

$$\begin{aligned} \mathbb{E}\phi_1(W^* - 1) &\leq C\mathbb{E}\phi_1\left(\left(\sum_{n=1}^{\infty}|D_n|^2\right)^{\frac{1}{2}}\right) \\ &\leq C\sum_{n=1}^{\infty}\mathbb{E}\phi_1(|D_n|), \end{aligned} \tag{5.12}$$

where $C = C(\phi_1) > 0$ is a constant depending only on ϕ_1.

Recalling that under $\mathbb{P}_{n-1}$, D_n is a sum of a sequence of martingale differences with respect to $(\tilde{\mathcal{F}}_k)$. Hence, again by the BDG-inequality applied to D_n, and the concavity of $\phi_1(x^{1/2})$, we get

$$\begin{aligned} \mathbb{E}_{n-1}\phi_1(|D_n|) &\leq C\mathbb{E}_{n-1}\phi_1\left(\left(\sum_{|u|=n-1}\frac{|\tilde{X}|^2}{\Pi_{n-1}^2}\right)^{\frac{1}{2}}\right) \\ &\leq C\mathbb{E}_{n-1}\sum_{|u|=n-1}\phi_1\left(\frac{|\tilde{X}|}{\Pi_{n-1}}\right) \\ &= CZ_{n-1} \cdot \mathbb{E}_{n-1}\phi_1\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}}\right) \end{aligned} \tag{5.13}$$

where $C > 0$ is independent of n. Taking integral on both sides of the inequality above, and noting that

ϕ_1' is concave, we obtain:

$$\begin{aligned}
\mathbb{E}\phi_1(|D_n|) &\leq C\mathbb{E}\mathbb{E}_\zeta\left(Z_{n-1}\cdot\mathbb{E}_{n-1}\phi_1\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}}\right)\right)\\
&= C\mathbb{E}\Pi_{n-1}\phi_1\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}}\right)\\
&= C\mathbb{E}|\tilde{X}_{n-1}|\int_0^1\phi_1'\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}}s\right)ds\\
&\leq C\mathbb{E}|\tilde{X}_{n-1}|\phi_1'(b^n|\tilde{X}_{n-1}|)\\
&= C\mathbb{E}|\tilde{X}|\phi_1'(b^n|\tilde{X}|), \qquad (5.14)
\end{aligned}$$

where $\tilde{X}$ is a random variable having the same distribution as $(\tilde{X}_n)_{n\geq 0}$ and $b=\mathbb{E}m_0^{-1}<1$. Similarly to (5.2), combining (5.12) and (5.14), we obtain

$$\begin{aligned}
\mathbb{E}\phi_1(W^*-1) &\leq C\mathbb{E}\sum_{n=1}^{\infty}|\tilde{X}|\phi_1'(b^n|\tilde{X}|)\\
&\leq C\mathbb{E}|\tilde{X}|\int_0^{|\tilde{X}|}\frac{\phi_1'(t)}{t}dt\\
&\leq C\mathbb{E}|\tilde{X}|(\tilde{\phi}_1(|\tilde{X}|)+1). \qquad (5.15)
\end{aligned}$$

As $\phi\asymp\phi_1$ and $\tilde{\phi}\asymp\tilde{\phi}_1$, this yields

$$\mathbb{E}\phi(W^*-1)\leq C\mathbb{E}|\tilde{X}|(\tilde{\phi}(|\tilde{X}|)+1)<\infty\,. \qquad (5.16)$$

Therefore $\mathbb{E}\phi(W^*)<\infty$, and

$$\mathbb{E}\phi(W)\leq\mathbb{E}\phi(W^*)<\infty\,.$$

□

6 Proof of Theorem 1.3

For the proof of Theorem 1.3, we shall need an extension of a theorem of Hsu and Robbins (1947) (see also Erdös (1949) or Baum and Katz (1965)). As usual, for a random variable X, we write $X^+=\max(X,0)$ and $X^-=\max(-X,0)$.

Lemma 6.1 *Let (X_i) be i.i.d. with $\mathbb{E}X_1\in[-\infty,\infty)$. If $\mathbb{E}(X_1^+)^2<\infty$, then for all $a>\mathbb{E}X_1$,*

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\sum_{i=1}^{n}X_i>na\right)<\infty\,. \qquad (6.1)$$

The result is due to Hsu and Robbins (1947) if $\mathbb{E}X_1^2<\infty$, and due to Heyde (1964, Theorem A; 1966, Theorem 1) if $\mathbb{E}|X_1|<\infty$. The present form is a consequence of Theorem 2.1 of Kesten and Maller (1996) which is more precise. As the theorem of Kesten and Maller is not easy to prove, for reader's convenience we give a short proof based of the theorem of Hsu and Robbins.

Proof of Lemma 6.1. Notice that for all $a,a_1,a_2\in\mathbb{R}$ with $a_1+a_2=a$,

$$\mathbb{P}\left(\sum_{i=1}^{n}X_i>na\right)\leq\mathbb{P}\left(\sum_{i=1}^{n}X_i^+>na_1\right)+\mathbb{P}\left(-\sum_{i=1}^{n}X_i^->na_2\right). \qquad (6.2)$$

By the theorem of Hsu and Robbins (1947), $\forall\ a_1>\mathbb{E}X_1^+$,

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\sum_{i=1}^{n}X_i^+>na_1\right)<\infty. \qquad (6.3)$$

Now for all $C > 0$, $\mathbb{P}\left(-\sum_{i=1}^n X_i^- > na_2\right) \leq \mathbb{P}\left(\sum_{i=1}^n -\min(X_i^-, C) > na_2\right)$. Therefore, again by the theorem of Hsu and Robbins (1947), $\forall\ a_2 > -\mathbb{E}\min(X_1^-, C)$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(-\sum_{i=1}^{n} X_i^- > na_2\right) < \infty. \tag{6.4}$$

Since $\lim_{C\to\infty} \mathbb{E}\min(X_1^-, C) = \mathbb{E}X_1^-$ by the monotone convergence theorem, (6.4) holds for all $a_2 > -\mathbb{E}X_1^-$. Together with (6.2) and (6.3), this implies that (6.1) holds for all $a > \mathbb{E}X_1^+ - \mathbb{E}X_1^- = \mathbb{E}X_1$. □

Proof of Theorem 1.3. Let

$$\ell(x) = \begin{cases} 1 - \frac{1}{2x}, & \text{if} \quad x > 1; \\ \frac{x}{2}, & \text{if} \quad x \leq 1. \end{cases}$$

Then $\phi(x) = x\ell(x)$ is convex, and the function $x \mapsto \phi(x^{1/2})$ is concave. By an argument similar to that in the proof of Theorem 1.2, we get (cf. (5.3))

$$\mathbb{E}\phi(W^* - 1) \leq C \sum_{n=1}^{\infty} \mathbb{E}|\tilde{X}|\ell\left(\frac{|\tilde{X}|}{\Pi_{n-1}}\right). \tag{6.5}$$

Let $b \in (e^{-\mathbb{E}\ln m_0}, 1)$ (by convention $e^{-\mathbb{E}\ln m_0} = 0$ if $\mathbb{E}\ln m_0 = +\infty$). For $n \geq 0$, we divide the domain of integration above into two parts according to $\{\Pi_n^{-1} \leq b^n\}$ or $\{\Pi_n^{-1} > b^n\}$, so that

$$\mathbb{E}\phi(W^* - 1) \leq C \sum_{n=0}^{\infty} (I_4(n) + I_4'(n)), \tag{6.6}$$

where

$$I_4(n) = \mathbb{E}|\tilde{X}|\ell(|\tilde{X}|\Pi_n^{-1})\mathbf{1}_{\{\Pi_n^{-1} \leq b^n\}},$$
$$I_4'(n) = \mathbb{E}|\tilde{X}|\ell(|\tilde{X}|\Pi_n^{-1})\mathbf{1}_{\{\Pi_n^{-1} > b^n\}}.$$

We first estimate $I_4(n)$. Noting that ℓ is increasing on $[0, \infty)$, we get $I_4(n) \leq \mathbb{E}|\tilde{X}|\ell(|\tilde{X}|b^n)$; moreover,

$$\begin{aligned} \sum_{n=0}^{\infty} I_4(n) &\leq C\mathbb{E}|\tilde{X}| \int_0^{|\tilde{X}|} \frac{\ell(t)}{t} dt \\ &\leq C\mathbb{E}|\tilde{X}|(1 + \ln^+ |\tilde{X}|) < \infty\,. \end{aligned} \tag{6.7}$$

To estimate $I_4'(n)$, as ℓ is bounded by 1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} I_4'(n) &\leq \mathbb{E}|\tilde{X}| \cdot \sum_{n=0}^{\infty} \mathbb{E}\mathbf{1}_{\{\Pi_n^{-1} > b^n\}} \\ &= \mathbb{E}|\tilde{X}| \cdot \sum_{n=0}^{\infty} \mathbb{P}(\Pi_n^{-1} > b^n). \end{aligned} \tag{6.8}$$

By Lemma 6.1, the sum on the right side of (6.8) is finite if $\mathbb{E}\left(\ln^+ \frac{1}{m_0}\right)^2 < \infty$. Therefore,

$$\mathbb{E}\phi(W^*) < \infty\,, \tag{6.9}$$

which is equivalent to $\mathbb{E}W^* < \infty$.

□

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